

A Remark on Normal Forms of Matrices

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INTRODUCTION

This remark answers the two problems raised in [2]. As in [4], we use the recent techniques of [3] and [5] of the representation theory of finite-dimensional algebras. It seems that these techniques provide methods of solution, as well as proper understanding, of such classification problems.

1. FIRST PROBLEM

The first problem of [2] asks for normal forms of $2m \times 2n$ complex matrices with respect to \mathbb{H} -similarity. Here, two complex $2m \times 2n$ matrices A, A' are said to be \mathbb{H} -similar if there exist formally quaternionic invertible (square) matrices P, Q such that $QA = A'P$, and a complex $2m \times 2n$ matrix P is called *formally quaternionic* if each block in its natural partition into 2×2 blocks has the form

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad \text{with } a, b \in \mathbb{C},$$

where \bar{c} denotes the complex conjugate of $c \in \mathbb{C}$.

In order to solve this problem, one may just follow the general procedure presented in [4] and illustrated there on the classification problem of $2m \times 2n$ real matrices with respect to \mathbb{C} -similarity. There, the problem was reformulated as the classification of real linear maps between two complex vector spaces. Similarly, in the present problem, we are concerned with the

classification of complex linear maps ψ between two quaternionic vector spaces $V_{\mathbb{H}}$ and $W_{\mathbb{H}}$. It is easy to see that this is equivalent to the classification of pairs of linear maps between two quaternionic spaces. Indeed, the \mathbb{C} -linear maps

$$\psi: V_{\mathbb{H}} \otimes_{\mathbb{H}} \mathbb{H}_{\mathbb{C}} \rightarrow W_{\mathbb{H}} \otimes_{\mathbb{H}} \mathbb{H}_{\mathbb{C}} \approx \text{Hom}_{\mathbb{H}}({}_{\mathbb{C}}\mathbb{H}_{\mathbb{H}}, W_{\mathbb{H}})$$

are in one-to-one correspondence with the \mathbb{H} -linear maps

$$\varphi: V_{\mathbb{H}} \otimes_{\mathbb{H}} \mathbb{H}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{H}_{\mathbb{H}} \rightarrow W_{\mathbb{H}},$$

and

$${}_{\mathbb{H}}\mathbb{H}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{H}_{\mathbb{H}} = {}_{\mathbb{H}}\mathbb{H}_{\mathbb{H}} \oplus {}_{\mathbb{H}}\mathbb{H}_{\mathbb{H}},$$

where the direct summands are generated by $1 \otimes 1 - j \otimes j$ and $(1 \otimes 1 + j \otimes j)i = 1 \otimes i - j \otimes k$ (here $1, i, j, k$ is the standard basis of $\mathbb{H}_{\mathbb{R}}$).

Thus, as in [4], we just translate the classification of the indecomposable representation of the species $\mathbb{H} \xrightarrow{\mathbb{H} \oplus \mathbb{H}} \mathbb{H}$ into matrix language (choosing suitable bases). Using the terminology and results of [3] and [5], we know that the subcategory of the homogeneous representations is the product of a uniserial category with one simple object and the (also uniserial) category of all finite-dimensional modules over the polynomial algebra $\mathbb{H}[x]$, and we can obtain the simple objects of these categories from the Addendum of [3]. Thus we get the following

THEOREM 1. *Every (nonzero) complex $2m \times 2n$ matrix is \mathbb{H} -similar to a zero-augmented product of matrices of the following types:*

- (i) $2(p+1) \times 2p$ matrices ($p = 1, 2, \dots$)

$$\begin{pmatrix} E_1 & & & & & & \\ E_{-1} & E_1 & & & & & 0 \\ & E_{-1} & E_1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ 0 & & & E_{-1} & E_1 & & \\ & & & & E_{-1} & & \end{pmatrix},$$

- (ii) the corresponding transposed $2p \times 2(p+1)$ matrices ($p=1,2,\dots$), and
 (iii) $2p \times 2p$ matrices ($p=1,2,\dots$)

$$\begin{bmatrix} E_c & E_1 & & & \\ & E_c & E_1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & E_c & E_1 \\ & & & & E_c \end{bmatrix} \quad \text{with } c = a + bi, \ a \geq 0,$$

or

$$\begin{bmatrix} E & E_1 & & & 0 \\ & E & E_1 & & \\ & & \ddots & \ddots & \\ 0 & & & E & E_1 \\ & & & & E \end{bmatrix},$$

where

$$E_c = \begin{pmatrix} 1-c & 0 \\ 0 & 1+\bar{c} \end{pmatrix} \quad \text{for complex } c \quad \text{and} \quad E = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

These matrices are \mathbb{H} -indecomposable (i.e. not \mathbb{H} -similar to a proper direct product of two matrices), and in the decomposition of a complex $2m \times 2n$ matrix, they are determined (up to their order) uniquely.

2. SECOND PROBLEM

The second problem asks for normal forms of $4m \times 4n$ real matrices with respect to \mathbb{H} -similarity. Here, two real $4m \times 4n$ real matrices A, A' are said to be \mathbb{H} -similar if there exist formally real-quaternionic invertible (square) matrices P, Q such that $QA = A'P$, and a real $4m \times 4n$ matrix P is called *formally real-quaternionic* if each block of its natural partition into 4×4

blocks has the form

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \quad \text{with } a, b, c, d \in \mathbb{R}.$$

Similarly to the first problem, the \mathbb{H} -similarity classes of real $4m \times 4n$ matrices correspond to the isomorphism classes of real linear maps between two quaternionic vector spaces, and again, since ${}_{\mathbb{H}}\mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}_{\mathbb{H}} \approx \mathbb{H}^4$, they correspond to the isomorphism classes of representations of the species $\mathbb{H} \rightarrow \mathbb{H}$. It is well known that this problem is "wild," so that one cannot expect a satisfactory normal form. For the benefit of the reader, we repeat here the argument.

Given any real $4m \times 4n$ matrix A , let

$$\text{End}_{\mathbb{H}}(A) = \{(P, Q) \mid P, Q \text{ formally real-quaternionic matrices with } QA = AP\}$$

be its \mathbb{H} -endomorphism ring (the ring operations are componentwise). It is clear that for \mathbb{H} -similar real $4m \times 4n$ matrices A, A' the \mathbb{R} -algebras $\text{End}_{\mathbb{H}}(A)$ and $\text{End}_{\mathbb{H}}(A')$ are isomorphic.

THEOREM 2. *For any finite dimensional \mathbb{R} -algebra R , there exists a real $4m \times 4n$ matrix A with $\text{End}_{\mathbb{H}}(A)$ isomorphic to R .*

Proof. Let R be generated by r_1, \dots, r_n . We will consider the left multiplication by r_i as an element of $\text{End}_{\mathbb{R}}(R)$, and denote it also by r_i . It is easy to check that the centralizer of the two $m \times m$ matrices

$$\alpha = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ 0 & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ r_1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & r_{n-1} & 1 & 0 \\ & & & r_n & 1 & 0 \end{pmatrix}$$

in $\text{End}_{\mathbb{R}}(X)$, with

$$X = \bigoplus_m R \quad \text{and} \quad m = n + 2,$$

is just the set of R -multiples of the identity, and thus isomorphic to R (see [1]). Consider now $Y_{\mathbb{H}} = X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}_{\mathbb{H}}$. The centralizer of $1 \otimes i$ and $1 \otimes j$ (where i, j denote the corresponding left multiplications on \mathbb{H}) in $\text{End}(Y_{\mathbb{H}})$ are the elements $\varphi \otimes 1$ with $\varphi \in \text{End}(X_{\mathbb{R}})$; thus the centralizer of $1 \otimes i$, $1 \otimes j$ and $\alpha \otimes 1 + \beta \otimes i$ in $\text{End}(Y_{\mathbb{H}})$ will be isomorphic to R . However, this centralizer is precisely the endomorphism ring $\text{End}_{\mathbb{H}}(A)$ of the real $4m \times 4m$ matrix A corresponding to the representation

$$\begin{array}{c} 1 \otimes 1 \\ 1 \otimes i \\ 1 \otimes j \\ (\alpha \otimes 1) + (\beta \otimes i) \end{array} \quad Y_{\mathbb{H}} \longrightarrow Y_{\mathbb{H}}$$

of $\mathbb{H} \xrightarrow{\mathbb{H}^4} \mathbb{H}$. ■

It follows from this theorem that a classification of real $4n \times 4n$ matrices with respect to \mathbb{H} -similarity is impossible: it would lead, at the same time, to a classification of all finite-dimensional \mathbb{R} -algebras.

3. CONCLUSION

Note that the first “open problem” is in fact, as we have shown above, a special case of the situation considered in the same paper [2]. It may perhaps be proper to emphasize two different objectives in dealing with classification problems: One is to find normal forms for a given problem; the other, usually easier objective is to show that two given problems have the same normal forms (modulo various discrete series of forms). The main theorem of [2] is a result of the second type (whereas the above solution of the first problem is of the first type). Let us remark that in such a situation, no simple normal form of matrices need exist at all—the classification of the similarity classes of matrices over a division ring seems to be a very difficult problem. On the other hand, the normal forms of matrices of discrete dimension type can always be listed [5], even in the “wild” situation of the second problem.

REFERENCES

- 1 S. Brenner, Decomposition properties of some small diagrams of modules, *Symposia Mathematica* 13:128–142 (1974).
- 2 D. Ž. Djoković, Classification of pairs of maps, *Linear Algebra and Appl.* 20:147–165 (1978).
- 3 V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* 173 (1976).
- 4 V. Dlab and C. M. Ringel, Normal forms of real matrices with respect to complex similarity, *Linear Algebra and Appl.* 17:107–124 (1977).
- 5 C. M. Ringel, Representations of K -species and bimodules, *J. Algebra* 41:269–302 (1976).

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